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Detours and paths: BRST complexes and worldline formalism

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Abstract: We construct detour complexes from the BRST quantization of worldline diffeomorphism invariant systems. This yields a method to efficiently extract physical quantum field theories from particle models with first class constraint algebras. As an example, we show how to obtain the Maxwell detour complex by gauging $\mathcal{N} = 2$ supersymmetric quantum mechanics in curved space. Then we concentrate on first class algebras belonging to a class of recently introduced orthosymplectic quantum mechanical models and give generating functions for detour complexes describing higher spins of arbitrary symmetry types. The first quantized approach facilitates quantum calculations and we employ it to compute the number of physical degrees of freedom associated to the second quantized, field theoretical actions.

Keywords: BRST Quantization, Gauge Symmetry, Sigma Models

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1 Introduction

Unquestionably, gauge theories are a central pillar of modern theoretical physics. Although they are usually presented in terms of a local symmetry of a field theoretic action principle, it is often useful to describe them in a first quantized language. The purpose of this Article is to present new tools to analyze gauge field theories using a first quantized picture, and to apply them to higher spin theories. These tools employ an algebraic construction known as the detour complex. The basic elements of the detour complex are differential operators which form the building blocks of the gauge theory under study. These differential operators can be represented as quantum mechanical operators. Crucially, by forming first class constraint algebras from them, one may consider *worldline* gauge theories. This procedure gives rise to a particle model which generically is diffeomorphism invariant on the worldline, and whose physical spectrum is related to the first quantization of the gauge field theory. Then, the BRST quantization of the particle naturally provides a cohomological complex, out of which one builds the detour complex. Equivalence of the BRST cohomology with the

detour cohomology guarantees that the correct physical information is properly encoded in the construction. From the detour complex one identifies an action principle for the gauge invariant field equations and may use the associated particle model to extract information about the quantized version of the theory.

Our first example — which provides much intuition — is the Maxwell detour. It describes the abelian gauge theory of differential forms and is related to the quantization of the $\mathcal{N} = 2$ spinning particle. We then consider detour complexes constructed out of the gauging of certain orthosymplectic quantum mechanical models [1], focusing mostly on symplectic subgroups. We use them to analyze the structure of gauge theories for bosonic fields of higher spins with arbitrary symmetry type [2] (see also [3–7]; we refer the reader also to the series of higher spin review Articles [8]).

Thus, in section 2 we present the structure and some elements of detour complexes. In section 3 we recall the use of path integrals to treat quantum field theories in first quantization. Section 4 contains the example of the Maxwell detour dealing with the abelian gauge symmetries of differential forms. In section 5 we construct detour complexes for higher spin fields of arbitrary symmetry type. Finally, we present our conclusions in section 6.

2 Detour complexes

As already mentioned, gauge theories are usually presented in terms of a local symmetry of an action principle. Other key ingredients, however, are gauge parameters, gauge fields, field equations and Bianchi identities. These can all be packaged in a single mathematical object known as a detour complex. Schematically

$$0 \longrightarrow \left\{\begin{smallmatrix}\text{gauge}\\ \text{parameters}\end{smallmatrix}\right\} \xrightarrow{\boldsymbol{d}} \left\{\begin{smallmatrix}\text{gauge}\\ \text{fields}\end{smallmatrix}\right\} \longrightarrow \cdots \qquad \cdots \longrightarrow \left\{\begin{smallmatrix}\text{field}\\ \text{equations}\end{smallmatrix}\right\} \xrightarrow{\boldsymbol{\delta}} \left\{\begin{smallmatrix}\text{Bianchi}\\ \text{identities}\end{smallmatrix}\right\} \longrightarrow 0 \tag{2.1}$$

$$\boldsymbol{G}$$

where for simplicity in the last two entries we have labeled the field space with the equations living on that space. Recall that a sequence of operators is called a complex when consecutive products vanish, so here

$$\boldsymbol{G}\boldsymbol{d} = 0 = \boldsymbol{\delta}\boldsymbol{G}\,. \tag{2.2}$$

These relations subsume the usual ones of gauge theories. Namely, if A is a gauge field, then $\boldsymbol{G}A = 0$ is its field equation, while if α is a gauge parameter, $A \to A + \boldsymbol{d}\alpha$ is the corresponding gauge transformation. The relation $\boldsymbol{G}\boldsymbol{d} = 0$ ensures that the field equations are gauge invariant.

We call $\boldsymbol{G}$ the detour or long operator since it connects dual complexes. Optimally, $\boldsymbol{G}$ is self-adjoint, so that one can use it to construct an action of the form $S = \frac{1}{2}\int(A, \boldsymbol{G}A)$ where the inner product is the one following naturally from the underlying first quantized quantum mechanical model. From the action principle viewpoint, since an arbitrary variation produces the field equations $\boldsymbol{G}A = 0$, specializing to a gauge transformation, the operator $\boldsymbol{\delta}$, dual to $\boldsymbol{d}$, must annihilate the equation of motion $\boldsymbol{\delta}\boldsymbol{G}A = 0$. This Noether-type identity generalizes the Bianchi identity for the Einstein tensor of general relativity, and is precisely the second relation above.

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The simplest example is the Maxwell detour. In that case gauge fields are one-forms, or sections of $\Lambda^1 M$, while the gauge parameters are zero-forms $\Lambda^0 M$. The field equations and Bianchi identities are also one and zero-forms, respectively. The differentials are the exterior derivative and codifferential while the detour operator is simply their product. In diagrammatic notation

$$0 \longrightarrow \wedge^0 M \xrightarrow{\boldsymbol{d}} \wedge^1 M \to \cdots \qquad \cdots \to \wedge^1 M \xrightarrow{\boldsymbol{\delta}} \wedge^0 M \longrightarrow 0 \tag{2.3}$$
$$\underbrace{\qquad\qquad\qquad\qquad}_{\boldsymbol{\delta d}}$$

Maxwell's equations are at once recognized as $\boldsymbol{\delta d} A = 0$, while gauge invariance and the Bianchi identity follow immediately from nilpotency of $\boldsymbol{d}$ and $\boldsymbol{\delta}$. The long operator $\boldsymbol{\delta d}$ connects the de Rham complex and its dual. Notice we could also not detour, and continue the de Rham complex, the next entry being two-forms, or in physics language, curvatures.

There are many other examples of detour complexes, for example: In four dimensions the long operator $\boldsymbol{\delta d}$ is conformally invariant. In six dimensions, there exists a higher order, conformally invariant detour operator $\boldsymbol{\delta}\Delta\boldsymbol{d} + \cdots$ [9, 10]. Another interesting variant is to twist the Maxwell complex by coupling to the Yang-Mills connection of a vector bundle over M. In this case, one obtains a complex exactly when the connection obeys the Yang-Mills equations [11, 12]. In this Article we will concentrate on de Rham detours and their generalization to "symmetric forms". The key idea is to use the relation between first quantized spinning particles and geometry. We obtain detour complexes by gauging these models and employing BRST quantization. In particular, the long operator connecting a complex with its dual, corresponds to a shift in the worldline diffeomorphism ghost number. This representation of the BRST cohomology in terms of field equations for gauge potentials is achieved by a careful choice of ghost polarizations. After reviewing the use of path integrals in first quantized approaches, we begin by studying generalized Maxwell complexes.

3 Path integrals

Just as for strings, also in particle theory a first quantized approach is often useful. While the field theory language is usually appropriate, the worldline approach can often be applied to more efficiently calculate quantum corrections, see [13] for a review. The simplest way to introduce the worldline formalism is perhaps to recall the example of a scalar field, whose free propagator and one-loop effective action can be represented in terms of a quantum mechanical hamiltonian supplemented by an integration over the Fock-Schwinger proper time. Much of the heat kernel literature can be classified under this point of view. An integration over the proper time signals that one is dealing with the quantum theory of a reparametrization invariant particle system; a relativistic spinless bosonic particle for the scalar field case. Similarly a spinning particle with $\mathcal{N} = 1$ supergravity on the worldline is related to the quantum theory of a Dirac field [14]. More generally $so(N)$ spinning particles are related to fields of spin $N/2$ [15]. Once the connection between reparametrization invariant particle models and quantum field theories is established, it is often advantageous to quantize the mechanical model with path integrals, i.e., summing over spinning

particle worldlines. For example, this worldline approach has been used for the $so(N)$ spinning particle in arbitrarily curved spaces with $N = 0, 1, 2$ to study the effective action for scalars [16], spin 1/2 [17], and arbitrary differential forms (including vectors) [18] coupled to gravity, respectively. The cases $N > 2$ do not admit a coupling to a generic curved space, but in [19] the worldline path integral has been considered in flat space (note that conformally flat backgrounds can be treated as well [20]) where the only information contained in the one-loop effective action is the number of circulating physical excitations.

Schematically, to compute the one-loop effective action Γ, one path integrates over closed worldlines with the topology of the circle S^1. Gauge fixing worldline diffeomorphisms produces an integral over the proper time β (the circumference of the circle). In arbitrary dimensions D, the result is

$$\Gamma = \int_{S^1} \mathcal{D}X \; e^{-S_{\text{particle}}[X]} = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta} \int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}}\, \gamma(x,\beta) \tag{3.1}$$

where the density $\gamma(x,\beta)$ can typically be computed in a small β expansion

$$\gamma(x,\beta) = a_0(x) + a_1(x)\beta + a_2(x)\beta^2 + \cdots \tag{3.2}$$

and $a_n(x)$ are called heat kernel coefficients. This expansion applies to massless particles and generically is not convergent in the upper β limit, even after renormalization. (Massive particles contain an extra factor $e^{-\frac{1}{2}m^2\beta}$ which improves the infrared behavior.) For a free theory in flat space, the only nonvanishing coefficient is the a_0, which is constant and counts the number of physical degrees of freedom circulating in the loop. This simplest of quantum computations is the one we focus on in this Article.

4 Maxwell detour

In this section we derive the Maxwell detour complex described in section 2. Our method relies on BRST quantization and a careful choice of ghost polarizations. We start by reviewing the underlying supersymmetric quantum mechanical model.

4.1 $\mathcal{N} = 2$ supersymmetric quantum mechanics

The physical Hilbert space of $\mathcal{N} = 2$ supersymmetric quantum mechanics is the space of differential forms $\Gamma(\wedge M)$ and, geometrically, its quantized Noether charges are the exterior derivative $\boldsymbol{d}$, codifferential $\boldsymbol{\delta}$, form Laplacian Δ and the degree operator $\mathbf{N}$ [21]. In this section, we review those results in detail.

The model is described by the action

$$S = \int dt \left\{ \frac{1}{2}\,\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu + i\bar{\psi}_\mu \frac{\nabla\psi^\mu}{dt} + \frac{1}{2}\,R_{\mu\nu\rho\sigma}\;\bar{\psi}^\mu\psi^\nu\bar{\psi}^\rho\psi^\sigma \right\}, \tag{4.1}$$

which is invariant under rigid $\mathcal{N} = 2$ supersymmetry $(\varepsilon, \bar{\varepsilon})$, U(1) fermion number symmetry (α) and worldline translations (ξ)

$$\begin{aligned} \delta x^\mu &= i\bar{\varepsilon}\psi^\mu + i\varepsilon\bar{\psi}^\mu + \xi\dot{x}^\mu\,, \\ \mathcal{D}\psi^\mu &= -\varepsilon\dot{x}^\mu + i\alpha\psi^\mu + \xi\frac{\nabla\psi^\mu}{dt}\,, \qquad \mathcal{D}\bar{\psi}^\mu = -\bar{\varepsilon}\dot{x}^\mu - i\alpha\psi^\mu + \xi\frac{\nabla\bar{\psi}^\mu}{dt}\,. \end{aligned} \tag{4.2}$$

In these formulæ $\mathcal{D}$ is the covariant variation: $\mathcal{D}X^\mu \equiv \delta X^\mu + \Gamma^\mu{}_{\nu\rho} X^\nu \delta x^\rho$ which obviates varying covariantly constant quantities. Invariance under supersymmetry follows easily upon noting the identity

$$\left[\mathcal{D}, \frac{\nabla}{dt}\right] X^\mu = \delta x^\rho \dot{x}^\sigma R_{\rho\sigma}{}^\mu{}_\nu X^\nu \,, \tag{4.3}$$

using which leaves only variations proportional to five fermions that vanish thanks to the second Bianchi identity for the Riemann tensor.

To quantize the model we work in a first order formulation

$$\dot{x}^\mu = \pi^\mu \,, \tag{4.4}$$

which follows from the action principle

$$S^{(1)} = \int dt \left\{ p_\mu \dot{x}^\mu + i\bar{\psi}_m \dot{\psi}^m - \frac{1}{2}\pi_\mu g^{\mu\nu} \pi_\nu + \frac{1}{2} R_{mnrs} \bar{\psi}^m \psi^n \bar{\psi}^r \psi^s \right\} . \tag{4.5}$$

Here we have used the vielbein $e_\mu{}^m$ to flatten the Lorentz indices on the fermions. Also the covariant and canonical momenta π_μ and p_μ are related by

$$\pi_\mu = p_\mu - i\omega_{\mu mn} \bar{\psi}^m \psi^n \,, \tag{4.6}$$

where $\omega_{\mu mn}$ is the spin connection. Since the symplectic current $p_\mu dx^\mu + i\bar{\psi}_m d\psi^m$ is expressed in Darboux coordinates, we immediately read off the quantum commutation relations

$$[p_\mu, x^\nu] = -i\delta^\nu_\mu \,, \qquad \{\bar{\psi}_m, \psi^n\} = \delta^n_m \,. \tag{4.7}$$

Motivated by geometry, we represent this algebra in terms of operators

$$p_\mu = \frac{1}{i}\frac{\partial}{\partial x^\mu} \,, \qquad \bar{\psi}_\mu = \frac{\partial}{\partial (dx^\mu)} \,, \qquad \psi^\mu = dx^\mu \,, \tag{4.8}$$

acting on wavefunctions

$$\Psi = \Psi(x, dx) = \sum_{k=0}^{D} F_{\mu_1 \ldots \mu_k}(x)\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} \,. \tag{4.9}$$

The variables dx^μ are Grassmann (so we will often denote their wedge products simply by juxtaposition) which means the coefficients $F_{\mu_1 \ldots \mu_k}$ in this expansion are totally antisymmetric, or in other words differential forms (or sections of $\wedge^k M$).

The Noether charges corresponding to supersymmetries $Q, \bar{Q}$, fermion number $\boldsymbol{N}$ and worldline translations H are now operators acting on wavefunctions. With a suitable normalization they equal

$$\begin{aligned} Q &= i\psi^\mu \pi_\mu \,, \\ \bar{Q} &= i\bar{\psi}^\mu \pi_\mu \,, \\ \boldsymbol{N} &= \psi^\mu \bar{\psi}_\mu, \\ H &= \frac{1}{2}\pi_m \pi^m - \frac{i}{2}\omega_m{}^{mn} \pi_n - \frac{1}{2} R_{\mu\nu\rho\sigma}\, \bar{\psi}^\mu \psi^\nu \bar{\psi}^\rho \psi^\sigma \,, \end{aligned} \tag{4.10}$$

and satisfy a $\mathcal{N}=2$ superalgebra

$$\{Q,\bar{Q}\}=-2H\,,\qquad [\boldsymbol{N},Q]=Q\,,\qquad [\boldsymbol{N},\bar{Q}]=-\bar{Q} \tag{4.11}$$

with all other (anti)commutators vanishing. These are quantum results, so their orderings matter and have been carefully arranged to (i) maintain the classical symmetry algebra and (ii) correspond to well known geometric operations. (This explains the explicit spin connection appearing in the Hamiltonian H.) In fact, the charges $(Q,\bar{Q},\boldsymbol{N},-2H)$ correspond precisely to the exterior derivative, codifferential, form degree and form Laplacian acting on differential forms

$$Q\Psi=\boldsymbol{d}\Psi\,,\qquad \bar{Q}\Psi=\boldsymbol{\delta}\Psi\,,\qquad -2H\Psi=\Delta\Psi\,. \tag{4.12}$$

Indeed the above superalgebra is precisely the usual set of relations for these operators

$$\begin{aligned} \boldsymbol{d}\boldsymbol{\delta}+\boldsymbol{\delta}\boldsymbol{d}=&\Delta\,, & \boldsymbol{d}\,\boldsymbol{N}=&(\boldsymbol{N}-1)\boldsymbol{d}\,, & \boldsymbol{\delta}\boldsymbol{N}=&(\boldsymbol{N}+1)\boldsymbol{\delta}\,,\\ \boldsymbol{d}\Delta=&\Delta\boldsymbol{d}\,, & \boldsymbol{\delta}\Delta=&\Delta\boldsymbol{\delta}\,, & \Delta\boldsymbol{N}=&\boldsymbol{N}\Delta\,. \end{aligned} \tag{4.13}$$

Our next task is to gauge this model and obtain a one-dimensional supergravity theory whose BRST quantization can then be studied.

4.2 $\mathcal{N}=2$ spinning particle

In Dirac quantization, gauging the supersymmetry and worldline translation symmetries of $\mathcal{N}=2$ supersymmetric quantum mechanics amounts to imposing constraints $Q=\bar{Q}=H=0$. This is implemented by lapse (alias worldline einbein) and gravitini Lagrange multipliers e, χ, $\bar{\chi}$ in the first order action

$$S^{(1)}=\int dt\left\{p_\mu\dot{x}^\mu+i\bar{\psi}_m\dot{\psi}^m-eH-\bar{\chi}Q-\chi\bar{Q}\right\}\,, \tag{4.14}$$

where $H=\frac{1}{2}\pi_\mu\pi^\mu-\frac{1}{2}\,R_{\mu\nu\rho\sigma}\;\bar{\psi}^\mu\psi^\nu\bar{\psi}^\rho\psi^\sigma$, $Q=i\psi^\mu\pi_\mu$, and $\bar{Q}=i\bar{\psi}^\mu\pi_\mu$ are now the classical analogues of (4.10). Integrating out the canonical momentum yields

$$S=\int dt\left\{\frac{1}{2e}\;\mathring{x}^{\,\mu}g_{\mu\nu}\;\mathring{x}^{\,\nu}+i\bar{\psi}_\mu\,\frac{\nabla\psi^\mu}{dt}+\frac{e}{2}\,R_{\mu\nu\rho\sigma}\;\bar{\psi}^\mu\psi^\nu\bar{\psi}^\rho\psi^\sigma\right\}\,, \tag{4.15}$$

where the supercovariant tangent vector

$$\mathring{x}^{\,\mu}\equiv\dot{x}^\mu-i\bar{\chi}\psi^\mu-i\chi\bar{\psi}^\mu\,. \tag{4.16}$$

The theory enjoys local supersymmetries

$$\begin{aligned} \delta x^\mu=&i\bar{\varepsilon}\psi^\mu+i\varepsilon\bar{\psi}^\mu\,, & \mathcal{D}\psi^\mu=&-\frac{1}{e}\;\mathring{x}^{\,\mu}\varepsilon\,, & \mathcal{D}\bar{\psi}^\mu=&-\frac{1}{e}\;\mathring{x}^{\,\mu}\bar{\varepsilon}\,,\\ \delta e=&2i\bar{\chi}\varepsilon+2i\chi\bar{\varepsilon}\,, & \delta\chi=&\dot{\varepsilon}\,, & \delta\bar{\chi}=&\dot{\bar{\varepsilon}}\,. \end{aligned} \tag{4.17}$$

Invariance is easily verified by noting that $\mathcal{D}\,\mathring{x}^{\,\mu}=i\bar{\varepsilon}\frac{\nabla\psi^\mu}{dt}+i\varepsilon\frac{\nabla\bar{\psi}^\mu}{dt}+\frac{\delta e\,\mathring{x}^{\,\mu}}{2e}$.

We must now BRST quantize the model. Firstly, notice that its Dirac quantization simply imposes the conditions

$$\boldsymbol{d}\Psi = \boldsymbol{\delta}\Psi = \Delta\Psi = 0 \tag{4.18}$$

on differential forms Ψ. This is the solution to the de Rham cohomology in terms of divergence-free, harmonic forms. These conditions are interpreted as the equations of motion for the degrees of freedom propagated by the $\mathcal{N} = 2$ spinning particle model in the target spacetime. They are given in terms of the k-form field strengths F_k for $k = 0, \ldots, D$ of (4.9) satisfying the Maxwell equations $\boldsymbol{d}F_k = \boldsymbol{\delta}F_k = 0$. It is well known that the corresponding number of physical degrees of freedom is given by DoF $= \sum_{k=0}^{D-2} \binom{D-2}{k} = 2^{D-2}$. However it is extremely useful to have a gauge theoretic description of these equations, for example by introducing auxiliary/gauge fields that can allow for a corresponding action principle. For this purpose we will employ the more powerful BRST quantization technique. As we shall see the above equations will correspond to the BRST cohomology at a given ghost number.

We proceed in a canonical framework and introduce fermionic worldline diffeomorphism ghosts with algebra

$$\{b, c\} = 1\,, \tag{4.19}$$

along with bosonic superghosts

$$[p, z^*] = 1 = [z, p^*]\,. \tag{4.20}$$

Wavefunctions in the BRST Hilbert space now depend also on (c, z^*, p^*)

$$\Psi = \Psi(x, dx, c, z^*, p^*) = \sum_{s,t=0}^{\infty} \frac{(z^*)^s (p^*)^t}{s!\; t!} \Big(\psi_{s,t} + c\,\chi_{s,t}\Big)\,, \tag{4.21}$$

where both $\psi_{s,t}$ and $\chi_{s,t}$ are sections of $\wedge M$ (ungraded differential forms). The choice to represent the BRST Hilbert space in terms of a Fock space with the above polarization is one of the key points of this Article. Although, one may *suspect* that the choice of polarization does not influence the cohomology of the BRST charge $Q_{\rm BRST}$, as we shall see, it has an extremely important impact on how that cohomology is *represented*. This point was first realized by Siegel, see [22]. In particular, we will find equations of motion expressed in terms of gauge potentials. These are realized by the the so-called long, or detour operator. For this it is crucial that we express BRST wavefunctions as an expansion in ghost number that is unbounded below and above. Only in this way, can we form a detour operator connecting de Rham and dual de Rham complexes.

On the BRST Hilbert space, it is easy to construct the nilpotent BRST charge, the result is

$$Q_{\rm BRST} = c\Delta + z^*\boldsymbol{\delta} + z\boldsymbol{d} - zz^*b\,. \tag{4.22}$$

The first three terms are the ghosts times the constraints while the final term reflects the first class constraint algebra $\{\boldsymbol{d}, \boldsymbol{\delta}\} = \Delta$. No further terms are necessary to ensure

$$Q^2_{\rm BRST} = 0\,, \tag{4.23}$$

as this algebra is rank 1. The other operator we shall need is the ghost number

$$N_{\rm gh} = cb + z^* p - p^* z \,, \tag{4.24}$$

which obeys

$$[N_{\rm gh}, Q_{\rm BRST}] = Q_{\rm BRST} \,. \tag{4.25}$$

Our task now is to compute the cohomology of $Q_{\rm BRST}$, namely

$$Q_{\rm BRST}\Psi = 0 \,, \qquad \Psi \sim \Psi + Q_{\rm BRST} X \,, \tag{4.26}$$

which is the topic of the next section.

4.3 BRST quantization

To solve the BRST cohomology,[1] we begin by requiring that Ψ is BRST closed. Computing $Q_{\rm BRST}\Psi$ we find the following conditions on the differential form-valued coefficients of the BRST wavefunction (4.21)

$$\begin{aligned}
\boldsymbol{d}\psi_{0,t+1} &= 0 \,, && t \geq 0 \,, \\
\chi_{s-1,t+1} &= \boldsymbol{\delta}\psi_{s-1,t} + \tfrac{1}{s}\boldsymbol{d}\psi_{s,t+1} \,, && t \geq 0 \,, \quad s \geq 1 \,, \\
\Delta\psi_{0,t} &= \boldsymbol{d}\chi_{0,t+1} \,, && t \geq 0 \,, \\
\Delta\psi_{s,t} &= s\boldsymbol{\delta}\chi_{s-1,t} + \boldsymbol{d}\chi_{s,t+1} \,, && t \geq 0 \,, \quad s \geq 1 \,.
\end{aligned} \tag{4.27}$$

The last pair of relations are actually not independent of the first pair save for the special case $t = 0$.

We may still shift Ψ by a BRST exact term $Q_{\rm BRST}X$, for which we make the ansatz

$$X = \sum_{s,t=0}^{\infty} \frac{(z^*)^s (p^*)^t}{s!\; t!} \Big(\alpha_{s,t} + c\, \beta_{s,t} \Big) \,. \tag{4.28}$$

Computing $Q_{\rm BRST}X$ we find equivalences/gauge invariances

$$\begin{aligned}
\psi_{0,t} &\sim \psi_{0,t} + \boldsymbol{d}\alpha_{0,t+1} \,, && t \geq 0 \,, \\
\psi_{s,t} &\sim \psi_{s,t} + \boldsymbol{d}\alpha_{s,t+1} + s(\boldsymbol{\delta}\alpha_{s-1,t} - \beta_{s-1,t+1}) \,, && t \geq 0 \,, \quad s \geq 1 \,, \\
\chi_{0,t} &\sim \chi_{0,t} + \Delta\alpha_{0,t} - \boldsymbol{d}\beta_{0,t+1} \,, && t \geq 0 \,, \\
\chi_{s,t} &\sim \chi_{s,t} + \Delta\alpha_{s,t} - s\boldsymbol{\delta}\beta_{s-1,t} - \boldsymbol{d}\beta_{s,t+1} \,, && t \geq 0 \,, \quad s \geq 1 \,.
\end{aligned} \tag{4.29}$$

To analyze these equations it helps to invoke the grading by ghost number. At a given ghost number $N_{\rm gh} = n$, there are an infinity of form fields

$$(\psi_{s,s-n}, \chi_{s,s-n+1}) \,, \tag{4.30}$$

indexed by s. However, using the closed conditions and exactness freedom, we can arrange for there to be only a single independent form field at each ghost number. To see this first examine the second equivalence relation in (4.29). By choice of $\beta_{s-1,t+1}$ we can set

$$\psi_{s,t} = 0 \,, \qquad t \geq 0 \,, s \geq 1 \,. \tag{4.31}$$

[1] We thank Andy Neitzke and Boris Pioline for an invaluable collaboration leading to the results of this section.

Substituting this choice in the closed conditions (4.27), we learn that

$$\begin{aligned} \chi_{s,t} &= 0\,, \qquad t \geq 1\,, \quad s \geq 1\,, \\ \chi_{0,t} &= \boldsymbol{\delta}\psi_{0,t-1}\,, \quad t \geq 1\,, \end{aligned} \tag{4.32}$$

so the remaining independent fields are $\psi_{0,t}$ with $t \geq 0$ and $\chi_{s,0}$ with $s \geq 0$. From (4.27) we see that they obey the closure conditions

$$\begin{aligned} \boldsymbol{d}\psi_{0,t} &= 0\,, \qquad t \geq 1\,, \\ \boldsymbol{\delta}\boldsymbol{d}\psi_{0,0} &= 0\,, \\ \boldsymbol{\delta}\chi_{s,0} &= 0\,, \qquad s \geq 0\,. \end{aligned} \tag{4.33}$$

The first and last of these are the closed conditions for the de Rham complex and its dual, while the middle relation is the detour operator.

Now we study exactness. Firstly we note that

$$\psi_{0,t} \sim \psi_{0,t} + \boldsymbol{d}\alpha_{0,t+1}\,, \quad t \geq 0\,, \tag{4.34}$$

whose interpretation in terms of de Rham complexes is clear. Then we observe that maintaining the gauge choice (4.31) means that further transformations must obey

$$\beta_{s,t} = \frac{1}{s+1}\boldsymbol{d}\alpha_{s+1,t} + \boldsymbol{\delta}\alpha_{s,t-1} \quad t \geq 1\,, \quad s \geq 0\,. \tag{4.35}$$

Employing this relation, then from (4.29) and using $\Delta = \boldsymbol{d}\boldsymbol{\delta} + \boldsymbol{\delta}\boldsymbol{d}$ we have

$$\chi_{00} \sim \chi_{00} + \boldsymbol{\delta}\boldsymbol{d}\alpha_{0,0} + \boldsymbol{d}(\boldsymbol{\delta}\alpha_{0,0} - \beta_{0,1}) = \chi_{0,0} + \boldsymbol{\delta}\boldsymbol{d}\alpha_{0,0} + \boldsymbol{d}(\boldsymbol{d}\alpha_{1,1})\,, \tag{4.36}$$

so that

$$\chi_{0,0} \sim \chi_{0,0} + \boldsymbol{\delta}\boldsymbol{d}\,\alpha_{0,0}\,. \tag{4.37}$$

Again this matches the detour operator. Finally a similar manipulation for the last equivalence in (4.29) for $t = 0$ and $s \geq 1$ yields

$$\chi_{s,0} \sim \chi_{s,0} + \boldsymbol{\delta}(-s\beta_{s-1,0} + \boldsymbol{d}\alpha_{s,0}) - \boldsymbol{d}(\beta_{s,1} - \boldsymbol{\delta}\alpha_{s,0})\,. \tag{4.38}$$

The last term vanishes using (4.35), so calling $\gamma_s \equiv -s\beta_{s-1,0} + \boldsymbol{d}\alpha_{s,0}$ $(s \geq 1)$, we find

$$\chi_{s,0} \sim \chi_{s,0} + \boldsymbol{\delta}\gamma_s\,, \qquad s \geq 1\,, \tag{4.39}$$

which matches perfectly the dual de Rham complex. Therefore we have proven the equivalence of the BRST cohomology and the Maxwell detour complex

$$\cdots \xrightarrow{\boldsymbol{d}} \wedge M \xrightarrow{\boldsymbol{d}} \wedge M \xrightarrow{\boldsymbol{d}} \wedge M \qquad \wedge M \xrightarrow{\boldsymbol{\delta}} \wedge M \xrightarrow{\boldsymbol{\delta}} \wedge M \xrightarrow{\boldsymbol{\delta}} \cdots \tag{4.40}$$

$$\boldsymbol{\delta}\boldsymbol{d}$$

The horizontal grading is by ghost number, increasing from left to right. The detour occurs at ghost number zero, at exactly which point the diffeomorphism ghost number makes its

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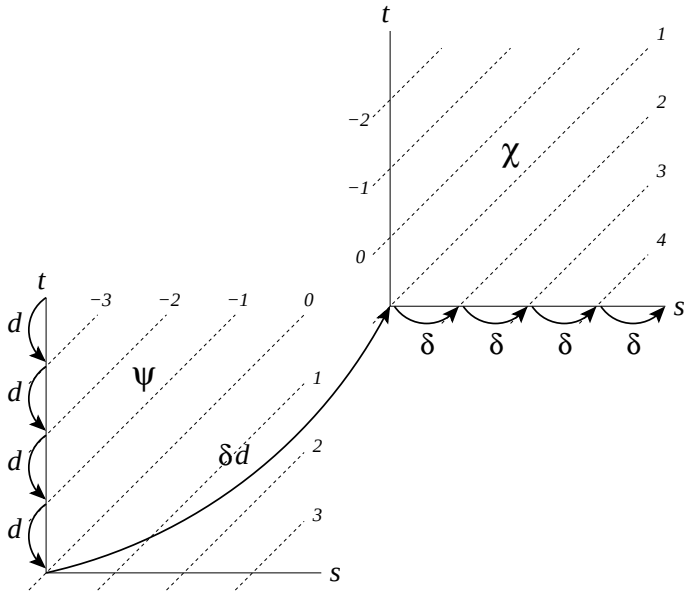

Figure 1. The Maxwell complex making its way through the BRST Hilbert space. Diagonal lines depict form fields of equal ghost number. The first graph plots fields $\psi_{s,t}$ and the second $\chi_{s,t}$.

jump by one unit. In figure 1 we depict the Maxwell detour complex snaking its way through the BRST Hilbert space.

The physical Hilbert space identified by the Dirac quantization procedure that we summarized earlier on is embedded in the BRST cohomology at fixed ghost number (i.e. zero ghost number for the present case). As we have seen the same cohomology is reproduced by the detour complex. An advantage is that the detour operator which acts at ghost number zero is formally self adjoint and can be used to construct a field theoretical gauge invariant action for the degrees of freedom propagated by the particle, that is $\int A\boldsymbol{\delta d}A \sim \int F^2$ as expected. The counting of degrees of freedom for this model is well known from standard work on antisymmetric tensor fields, and we can reproduce it using the first quantized picture in a somewhat simpler way.

4.4 Counting degrees of freedom

To extract quantum information one can equivalently use either the first quantized picture of the $\mathcal{N} = 2$ spinning particle or the second quantized, gauge invariant, field theory action. The first quantized approach is quite efficient, and we use it here to compute the number of physical degrees of freedom.

We need to evaluate the partition function of the $\mathcal{N} = 2$ spinning particle on the circle to get the one-loop effective action Γ. With euclidean conventions it reads

$$\Gamma[g_{\mu\nu}] = \int_{S^1} \frac{\mathcal{D}X\,\mathcal{D}G}{\text{Vol(Gauge)}}\, e^{-S[X,G;g_{\mu\nu}]}\,, \tag{4.41}$$

where $X = (x^\mu, \psi^\mu, \bar{\psi}^\mu)$ and $G = (e, \chi, \bar{\chi})$ indicate the fields that must be integrated over.

$S[X, G; g_{\mu\nu}]$ is the euclidean version of the action in (4.15)

$$S[X, G; g_{\mu\nu}] = \int_0^1 d\tau \left\{ \frac{1}{2e} \overset{\circ}{x}{}^\mu g_{\mu\nu} \overset{\circ}{x}{}^\nu + i\bar{\psi}_\mu \frac{\nabla\psi^\mu}{dt} - \frac{e}{2} R_{\mu\nu\rho\sigma}\ \bar{\psi}^\mu\psi^\nu\bar{\psi}^\rho\psi^\sigma \right\} . \tag{4.42}$$

The division by the volume of the gauge group implies that we need to fix the gauge symmetries. The loop (i.e. the circle S^1) is described by taking the euclidean time $\tau \in [0, 1]$, imposing periodic boundary conditions on the bosons (x^μ, e) and antiperiodic boundary conditions on the fermions $(\psi^\mu, \bar{\psi}^\mu, \chi, \bar{\chi})$. The gauge symmetries can be used to fix the supergravity multiplet to $\hat{G} = (\beta, 0, 0)$, where β is the leftover modulus that must be integrated over, i.e. the proper time. As the gravitini χ and $\bar{\chi}$ are antiperiodic, their susy transformations (4.17) are invertible so that they can be completely gauged away, leaving Faddeev-Popov determinants and no moduli. This produces

$$\Gamma[g_{\mu\nu}] = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta} \left(\mathrm{Det}_{_A}\partial_\tau\right)^{-2} \int_{S^1} \mathcal{D}X\, e^{-S[X,\hat{G};g_{\mu\nu}]} , \tag{4.43}$$

where the proper time measure takes into account the effect of the symmetry generated by the Killing vector on the circle. Note that the Faddeev-Popov determinants with antiperiodic boundary conditions (denoted by the subscript A) coming from the local supersymmetry do not depend on the target space geometry. The overall normalization $(-1/2)$ has been inserted to match with the standard result for a single real scalar particle.

We are interested in computing the number of physical degrees of freedom, so we lose no generality[2] by taking the flat limit $g_{\mu\nu} = \delta_{\mu\nu}$, and evaluate the remaining free path integral over the coordinates x^μ and their fermionic partner $\psi^\mu, \bar{\psi}^\mu$

$$\Gamma[\delta_{\mu\nu}] = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta} \left(\mathrm{Det}_{_A}\partial_\tau\right)^{D-2} \int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}} . \tag{4.44}$$

Apart from the standard volume term and the correctly normalized proper time factors, this result contains the degrees of freedom propagating in the loop,

$$\mathrm{DoF} = (\mathrm{Det}_{_A}\partial_\tau)^{D-2} . \tag{4.45}$$

The free fermionic determinant is easily computed: the antiperiodic boundary conditions produce a trace over the corresponding two-dimensional Hilbert space. Thus $\mathrm{Det}_{_A}\partial_\tau = 2$ and the degrees of freedom $\mathrm{DoF} = 2^{D-2}$ as expected. This first quantized picture has been used quite extensively in [18] to describe the quantum properties of the gauge theory of differential forms coupled to gravity.

5 Mixed higher spin detour

In [1], supersymmetric quantum mechanical models were constructed with R-symmetries obeying the superalgebra $osp(Q|2r)$. The "supercharges" of these models transformed under the fundamental representation of $osp(Q|2r)$, and therefore generated Q Grassmann

[2] For partially massless theories [23], more care is needed because there are various massless limits, but here we are only interested in the strictly massless one.

odd supersymmetries and $2r$ Grassmann even symmetries. The models were constructed in curved backgrounds. It was found that the supercharges commuted with the Hamiltonian only if the background manifold was a locally symmetric space, at general values of (Q, r). For low-lying values of (Q, r) the models coincide with well known quantum mechanical theories. The $osp(1,0)$ models is the $\mathcal{N} = 1$ supersymmetric quantum mechanics whose single supercharge corresponds to the Dirac operator and whose Hilbert space describes spinors. The $osp(2|0)$ theory reproduces the $\mathcal{N} = 2$ supersymmetric quantum mechanics described in section 4. For both those models the locally symmetric space condition is not necessary. In this section we concentrate on the models with R-symmetry $osp(0|2r) = sp(2r)$. These models are purely bosonic. Their Hilbert spaces correspond to multi-symmetric forms and their quantized Noether charges yield symmetrized gradient and divergence type-operators. We will concentrate on the simplest locally symmetric space-Minkowski space, although many of our computations should generalize easily at least to constant curvature spaces. We begin with the simplest $sp(2)$ model, which describes totally symmetric tensors or "symmetric forms".

5.1 Symmetric forms and $sp(2)$ quantum mechanics

Symmetric forms share many similarities with their totally anti-symmetric counterparts-differential forms. They are expressed in terms of totally symmetric tensors and *commuting* coordinate differentials so that a symmetric rank s tensor $\varphi_{(\mu_1 \dots \mu_s)}$ becomes

$$\Phi = \varphi_{\mu_1 \dots \mu_s} dx^{\mu_1} \cdots dx^{\mu_s} \,. \tag{5.1}$$

There is an algebra of operations — gradient, divergence, metric, trace and the wave operator first introduced by Lichnerowicz [24] and systemized in [1, 25, 26] (see [4, 27–29] for other studies) — which greatly facilitates computations when the rank s is large or even arbitrary. In particular, it is important to note that in this algebra (just as for differential forms) it is no longer forbidden to add tensors of different ranks.

In flat space, there are six distinguished operators mapping symmetric tensors to symmetric tensors:

N –Counts the number of indices

$$\mathbf{N}\ \Phi = s\Phi \,. \tag{5.2}$$

tr –Traces over a pair of indices

$$\mathbf{tr}\ \Phi = s(s-1)\varphi^{\rho}{}_{\rho\mu_3 \dots \mu_s} dx^{\mu_3} \cdots dx^{\mu_s} \,. \tag{5.3}$$

g –Adds a pair of indices using the metric

$$\mathbf{g}\ \Phi = g_{\mu_1\mu_2} \varphi_{\mu_3 \dots \mu_{s+2}} dx^{\mu_1} \cdots dx^{\mu_{s+2}} \,. \tag{5.4}$$

div –The symmetrized divergence

$$\mathbf{div}\ \Phi = s\nabla^{\rho} \varphi_{\rho\mu_2 \dots \mu_s} dx^{\mu_2} \cdots dx^{\mu_s} \,. \tag{5.5}$$

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grad –The symmetrized gradient

$$\mathbf{grad}\ \Phi = \nabla_{\mu_1}\varphi_{\mu_2\ldots\mu_{s+1}}dx^{\mu_1}\cdots dx^{\mu_{s+1}}\,. \tag{5.6}$$

Δ –The Bochner Laplacian

$$\Delta = \nabla^{\mu}\nabla_{\mu}\,. \tag{5.7}$$

The calculational advantage of these operators is the algebra they obey

$$\begin{aligned}
[\mathbf{N},\mathbf{tr}] &= -2\mathbf{tr}\,, \quad [\mathbf{N},\mathbf{div}] = -\mathbf{div}\,, \quad [\mathbf{N},\mathbf{grad}] = \mathbf{grad}\,, \quad [\mathbf{N},\mathbf{g}] = 2\mathbf{g}\,,\\
[\mathbf{tr},\mathbf{grad}] &= 2\mathbf{div}\,, \quad [\mathbf{tr},\mathbf{g}] = 4\mathbf{N}+2D\,, \quad [\mathbf{div},\mathbf{g}] = 2\mathbf{grad}\,,\\
[\mathbf{div},\mathbf{grad}] &= \Delta\,.
\end{aligned} \tag{5.8}$$

All other commutators vanish.

Symmetric forms may be interpreted as the Hilbert space of a quantum mechanical model whose quantum Noether charges are given by the operators above [1]. For flat backgrounds this theory is described by the simple action

$$S = \int dt\left\{\frac{1}{2}\dot{x}^{\mu}\dot{x}_{\mu} + iz^{*}_{\mu}\dot{z}^{\mu}\right\}\,. \tag{5.9}$$

Here the complex variables (z^{μ}, z^{*}_{μ}) are viewed as oscillator degrees of freedom describing the index-structure of wave functions so that upon quantization

$$z^{*\mu} \mapsto dx^{\mu}\,, \qquad z_{\mu} \mapsto \frac{\partial}{\partial(dx^{\mu})}\,. \tag{5.10}$$

The model's symmetries, Noether charges and their relation to the geometric operators given above is described in detail in [1]. In particular, the $sp(2)$ "R-symmetry" is generated by the triple $(\mathbf{tr}, \mathbf{N}+\frac{D}{2}, \mathbf{g})$. The pair of operators $\{\mathbf{div},\mathbf{grad}\}$ transform as a doublet under this $sp(2)$ and their commutator produces the Hamiltonian which corresponds to the Laplacian Δ. In this sense, $\{\mathbf{div},\mathbf{grad}\}$ could be viewed as pair of "bosonic supercharges".

Totally symmetric tensor higher spin theories can be formulated in terms of the algebra (5.8), so this relationship between those operators and the quantum mechanical model (5.9) yields a first-quantized worldline approach to these models. From that viewpoint we need to construct a spinning particle model by gauging an appropriate set of symmetries. The BRST cohomology of that one dimensional gauge theory then yields the physical spectrum of a higher spin field theory. We construct this spinning particle model in section 5.5. When the symmetries that we choose to gauge form a Lie algebra, the BRST problem becomes equivalent to one in Lie algebra cohomology. This is the topic of our next section.

5.2 BRST quantization and lie algebra cohomology

In this section we formulate the theory of massless, totally symmetric higher spins [33, 34] as the Lie algebra cohomology of a very simple algebra $\mathfrak{g}$:

$$\mathfrak{g}_A = \Big\{\mathbf{tr}\,,\ \mathbf{div}\,,\ \mathbf{grad}\,,\ \Delta\Big\}\,, \tag{5.11}$$

acting on the vector space V of symmetric forms. The BRST quantization for this algebra was first studied in [30] (see also [31, 32]).

Let us very briefly review the relationship between BRST quantization and Lie algebra cohomology. In BRST quantization the BRST Hilbert space is expressed as wavefunctions expanded in anticommuting ghosts

$$\Psi_{\rm BRST} = \sum_{k=0}^{\dim \mathfrak{g}} c^{A_1} \cdots c^{A_k} \Psi_{A_1 \ldots A_k} \,, \tag{5.12}$$

while in Lie algebra cohomology the V-valued wavefunctions $\Psi_{A_1 \ldots A_k}$ are viewed as multilinear maps $\mathfrak{g}^{\wedge k} \to V$ and form the cochains of a complex. The cochain degree k is BRST ghost number. The Chevalley-Eilenberg differential δ [35]

$$\Psi_{A_1 \ldots A_k} \stackrel{\delta}{\to} \mathfrak{g}_{[A_1} \Psi_{A_2 \ldots A_{k+1}]} - \frac{k}{2} f^B_{[A_1 A_2} \Psi_{|B| A_3 \ldots A_{k+1}]} \,, \tag{5.13}$$

can be compactly expressed in terms of the BRST charge

$$Q_{\rm BRST} = c^A \mathfrak{g}_A - \frac{1}{2} f^C_{AB} c^A c^B \frac{\partial}{\partial c^C} \,, \tag{5.14}$$

acting at ghost number k. The cohomology of $Q_{\rm BRST}$ at ghost number k equals the Lie algebra cohomology $H^k(\mathfrak{g}, V)$.

Returning to our specific Lie Algebra (5.11), we now relate its Lie algebra cohomology at degree one to the massless higher spin theory. At degree one, our problem is a very simple one: We first introduce a wavefunction for every generator

$$\Psi_A = \left\{ \Psi_{\mathbf{tr}}, \Psi_{\mathbf{div}}, \Psi_{\mathbf{grad}}, \Psi_\Delta \right\} . \tag{5.15}$$

The closure condition $\Psi_A \in \ker\delta$ yields a set of $\binom{\dim \mathfrak{g}}{2}$ differential equations following directly from the commutation relations (5.8)

$$\begin{aligned}
\mathbf{tr}\ \Psi_{\mathbf{div}} - \mathbf{div}\ \Psi_{\mathbf{tr}} &= 0 \,, \\
\mathbf{tr}\ \Psi_{\mathbf{grad}} - \mathbf{grad}\ \Psi_{\mathbf{tr}} &= 2\, \Psi_{\mathbf{div}} \,, \\
\mathbf{tr}\ \Psi_\Delta - \Delta\ \Psi_{\mathbf{tr}} &= 0 \,, \\
\mathbf{div}\ \Psi_{\mathbf{grad}} - \mathbf{grad}\ \Psi_{\mathbf{div}} &= \Psi_\Delta \,, \\
\mathbf{div}\ \Psi_\Delta - \Delta\ \Psi_{\mathbf{div}} &= 0 \,, \\
\mathbf{grad}\ \Psi_\Delta - \Delta\ \Psi_{\mathbf{grad}} &= 0 \,.
\end{aligned} \tag{5.16}$$

Exactness, $\Psi_A \sim \Psi_A + X_A$ with $X_A \in \mathrm{im}\delta$, yields the gauge invariances of this set of equations

$$\begin{aligned}
\delta\Psi_{\mathbf{tr}} &= \mathbf{tr}\ \xi \,, \\
\delta\Psi_{\mathbf{div}} &= \mathbf{div}\ \xi \,, \\
\delta\Psi_{\mathbf{grad}} &= \mathbf{grad}\ \xi \,, \\
\delta\Psi_\Delta &= \Delta\ \xi \,.
\end{aligned} \tag{5.17}$$

The fields $(\Psi_\Delta, \Psi_{\mathbf{grad}}, \Psi_{\mathbf{div}})$ correspond to the BRST triplet structure discussed in [31] (denoted (C, φ, D)) while $\Psi_{\mathbf{tr}}$ is the compensator field introduced there.

Eliminating the fields $\Psi_{\mathbf{div}}$ and Ψ_Δ using the second and fourth equations in (5.16), we find that only the first and sixth of these equations are independent and give a description of all massless, totally symmetric higher spins in terms of a pair of unconstrained fields $(\Psi_{\mathbf{grad}}, \Psi_{\mathbf{tr}})$ with a single unconstrained gauge parameter ξ

$$\begin{aligned} \mathbf{G}\ \Psi_{\mathbf{grad}} &= \frac{1}{2}\ \mathbf{grad}^3\ \Psi_{\mathbf{tr}}\,, \\ \mathbf{tr}^2\ \Psi_{\mathbf{grad}} &= 4\left(\mathbf{div} + \frac{1}{4}\mathbf{grad}\ \mathbf{tr}\right)\ \Psi_{\mathbf{tr}}\,, \\ \delta\Psi_{\mathbf{grad}} &= \mathbf{grad}\ \xi\,, \\ \delta\Psi_{\mathbf{tr}} &= \mathbf{tr}\ \xi\,. \end{aligned} \tag{5.18}$$

Here the operator $\mathbf{G}$ is given by

$$\mathbf{G} = \Delta - \mathbf{grad}\ \mathbf{div} + \frac{1}{2}\ \mathbf{grad}^2\ \mathbf{tr}\,. \tag{5.19}$$

Although, for some contexts, a formulation of higher spin dynamics in terms of unconstrained fields can be useful, the above system has the disadvantage that it has terms cubic in derivatives. This problem can be removed by using some of the gauge freedom to set the field $\Psi_{\mathbf{tr}} = 0$. This yields the standard description of massless, totally symmetric higher spins in terms of a doubly trace-free symmetric tensor and a trace-free gauge parameter

$$\begin{aligned} \mathbf{G}\ \Psi_{\mathbf{grad}} &= 0 = \mathbf{tr}^2\ \Psi_{\mathbf{grad}}\,, \\ \delta\Psi_{\mathbf{grad}} &= \mathbf{grad}\ \xi\,, \qquad\qquad \mathbf{tr}\ \xi = 0\,. \end{aligned} \tag{5.20}$$

It is important to note that since we kept the rank s, or in other words the eigenvalue of the index operator $\mathbf{N}$, arbitrary these relations generate the gauge invariant equations of motion for fields of any spin. We also remark that the operators $\{\mathbf{G},\ \mathbf{tr}^2\}$ themselves generate a first class algebra.

Finally, we can formulate this system in terms of a detour complex as follows. The field equation $\mathbf{G}\ \Psi_{\mathbf{grad}} = 0$ is equivalent to the equation

$$\mathcal{G}\ \Psi_{\mathbf{grad}} \equiv \left(1 - \frac{1}{4}\ \mathbf{g}\ \mathbf{tr}\right)\ \mathbf{G}\ \Psi_{\mathbf{grad}} = 0\,, \tag{5.21}$$

where

$$\mathcal{G} = \Delta - \mathbf{grad}\ \mathbf{div} + \frac{1}{2}\ \left(\mathbf{grad}^2\ \mathbf{tr} + \mathbf{g}\ \mathbf{div}^2\right) - \frac{1}{4}\ \mathbf{g}\ \left(2\Delta + \mathbf{grad}\ \mathbf{div}\right)\ \mathbf{tr}\,. \tag{5.22}$$

We will call this operator the higher spin Einstein operator, since if $\Psi_{\mathbf{grad}} = h_{\mu\nu}dx^\mu dx^\nu$ it then produces the linearized Einstein tensor

$$\mathcal{G}\Psi_{\mathbf{grad}} = (\Delta h_{\mu\nu} - 2\nabla_\mu\nabla^\rho h_{\rho\nu} + \nabla_\mu\nabla_\nu h^\rho_\rho + \eta_{\mu\nu}\nabla^\rho\nabla^\sigma h_{\rho\sigma} - \eta_{\mu\nu}\Delta h_\rho{}^\rho)dx^\mu dx^\nu\,. \tag{5.23}$$

The higher spin Einstein operator obeys identities

$$\mathbf{div}\ \mathcal{G} = 0 \qquad \mathrm{mod}_{\mathrm{left}}\ \mathbf{g}\,, \tag{5.24}$$

$$\mathcal{G}\ \mathbf{grad} = 0 \qquad \mathrm{mod}_{\mathrm{right}}\ \mathbf{tr}\,, \tag{5.25}$$

where equality holds up to terms proportional to the operators $\mathbf{g}$ and $\mathbf{tr}$ acting from the left and right, respectively. This allows us to form a detour complex

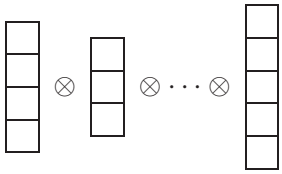
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \odot TM/_{\mathbf{tr}} & \xrightarrow{\mathbf{grad}} & \odot TM/_{\mathbf{tr}^2} & \to \cdots & \cdots \to & \odot TM/_{\mathbf{tr}^2} & \xrightarrow{\mathbf{div}} \odot TM/_{\mathbf{tr}^2} \longrightarrow 0 \\ & & & & \downarrow & & & \uparrow & \\ & & & & \multicolumn{3}{c}{\mathcal{G}} & & \end{array} \tag{5.26}$$

where $\odot TM/\bullet$ denotes symmetric tensors modulo the relation $\bullet$. The operator $\mathcal{G}$ is formally self-adjoint so the equation of motion $\mathcal{G}\Psi_{\mathbf{grad}} = 0$ comes from an action principle $S = \frac{1}{2}\int(\Phi, \mathcal{G}\Phi)$ where the inner product $(\cdot,\cdot)$ is the one inherited from the underlying quantum mechanical model. The relations (5.24) and (5.25) express the Bianchi identity and gauge invariance of this field equation. Our next task is to generalize this construction to tensors of arbitrary symmetry types, and in particular find compact expressions for the Einstein operators for these theories.

5.3 Mixed tensors and $sp(2r)$ quantum mechanics

Tensors transforming under arbitrary representations of $gl(D)$ can be expressed either in terms of:

1. Tensors labeled by groups of antisymmetric indices $\omega_{[\mu_1^1\ldots\mu_{k_1}^1]\cdots[\mu_1^s\ldots\mu_{k_s}^s]}$ —"multi-forms"— or schematically

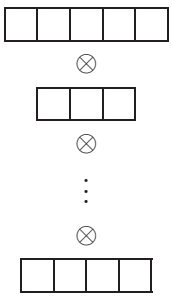


(5.27)

where s labels the number of antisymmetric columns, while the k_i label the number of boxes in each column.

or

2. Tensors labeled by groups of symmetric indices $\varphi_{(\mu_1^1\ldots\mu_{s_1}^1)\cdots(\mu_1^r\ldots\mu_{s_r}^r)}$ —"multi-symmetric forms"— or schematically



(5.28)

where r labels the number of symmetric rows, while the s_i label the number of boxes in each row.

In each case irreducible $gl(D)$ representations are obtained by placing algebraic constraints on tensors of these types akin to the Bianchi identity of the first kind obeyed by the Riemann tensor. Supersymmetric quantum mechanical models whose Hilbert spaces are populated by the tensors described above were developed in [1]. The R symmetry groups are $O(2s)$ and $\mathrm{Sp}(2r)$ for the two respective cases. Models for tensors with both symmetric and antisymmetric groups of indices also exist and have an $osp(2s|2r)$ R-symmetry. The $\mathcal{N}=2$ and $sp(2)$ (super)symmetric quantum mechanical models described above are the lowest lying examples of these.

Although, all the computations in this work, should in principle carry over to both $osp(Q|2r)$ models for arbitrary integers[3] (Q,r), here we concentrate on the $sp(2r)$ case. There are two reasons for this choice. Firstly, because the rows in (5.28) are symmetric, this allows us to handle arbitrarily high spins without introducing arbitrarily quantum mechanical oscillator modes. In particular, if we take $r \geq D$ wavefunctions span tensors of arbitrary type. Secondly, since only bosonic oscillators are required, the BRST ghosts will all be fermionic leading to a BRST wavefunction with a finite expansion in ghost modes and in turn a Lie algebra, rather than Lie superalgebra cohomology problem.

The quantum mechanical model whose wavefunctions describe multi-symmetric forms derives from the simple action principle

$$S = \int dt \left\{ \frac{1}{2}\dot{x}^\mu \dot{x}_\mu + i z^*_{i\mu} \dot{z}^{i\mu} \right\} . \tag{5.29}$$

Here we have introduced $2r$ oscillators $(z^*_{i\mu}, z^{i\mu})$ with $i = 1 \ldots r$. Their kinetic term can be written in the manifestly $sp(2r)$ symmetric way $\frac{i}{2} z_{\alpha\mu} \epsilon^{\alpha\beta} \dot{z}^\mu_\beta$ where $z_\alpha = (z^*_i, z^i)$ and $\epsilon^{\alpha\beta}$ is the antisymmetric invariant tensor of $sp(2r)$. Again, this theory can be coupled to curved backgrounds; we refer to [1, 36] for details.

Upon quantization the oscillators can be represented in terms of sets of commuting differentials [1, 6]

$$z^{*\mu}_i = d_i x^\mu , \qquad z^i_\mu = \frac{\partial}{\partial(d_i x^\mu)} . \tag{5.30}$$

The tensor depicted in (5.28) is then denoted

$$\Phi = \varphi_{(\mu^1_1 \ldots \mu^1_{s_1}) \cdots (\mu^k_1 \ldots \mu^r_{s_r})} d_1 x^{\mu^1_1} \cdots d_1 x^{\mu^1_{s_1}} \ \cdots \ d_r x^{\mu^r_1} \cdots d_r x^{\mu^r_{s_r}} . \tag{5.31}$$

The $sp(2)$ generators $\mathbf{g}$, $\mathbf{N}$ and $\mathbf{tr}$ of the above sections are promoted to $r \times r$ matrices of operators:

$$\begin{aligned} \mathbf{g}_{ij} &= d_i x^\mu g_{\mu\nu} d_j x^\nu , \\ \mathbf{N}^j_i &= dx^\mu_i \frac{\partial}{\partial(d_j x^\mu)}, \\ \mathbf{tr}^{ij} &= \frac{\partial}{\partial(d_i x^\mu)} g^{\mu\nu} \frac{\partial}{\partial(d_j x^\nu)} . \end{aligned} \tag{5.32}$$

[3] Note that when Q is odd, the model includes spinor fields [1].

The operators $(\mathbf{g}_{ij}, \mathbf{N}_i^j, \mathbf{tr}^{ij})$ generate $sp(2r)$

$$\begin{aligned}
[\mathbf{N}_i^j, \mathbf{g}_{kl}] &= 2\delta^j_{(k}\mathbf{g}_{l)i}\,, \\
[\mathbf{tr}^{ij}, \mathbf{g}_{kl}] &= 4\delta^{(i}_{(k}\mathbf{N}^{j)}_{l)} + 2D\,\delta^i_{(k}\delta^j_{l)}\,, \\
[\mathbf{N}_i^j, \mathbf{tr}^{kl}] &= -2\delta_i^{(k}\mathbf{tr}^{l)j}\,, \\
[\mathbf{N}_i^j, \mathbf{N}_k^l] &= \delta_k^j\mathbf{N}_i^l - \delta_i^l\mathbf{N}_j^k\,.
\end{aligned} \tag{5.33}$$

They correspond to "R-symmetries" of the model (5.29). Geometrically, in terms of the picture (5.28), they count the number of indices in a given row, move boxes from one row to another, and add or remove pairs of boxes to or from (possibly distinct) rows using the metric tensor.

The differential operators **div** and **grad** are replaced by $2r$ operators corresponding to the divergence and gradient acting on each row in (5.28). These form the fundamental representation of $sp(2r)$.

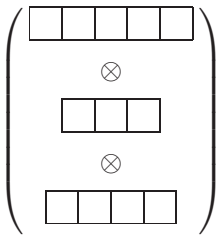
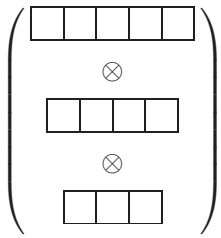
$$\begin{aligned}
[\mathbf{N}_i^j, \mathbf{grad}_k] &= \delta_k^j\ \mathbf{grad}_i\,, & [\mathbf{N}_i^j, \mathbf{div}^k] &= -\,\delta_i^k\ \mathbf{div}^j\,, \\
[\mathbf{tr}^{ij}, \mathbf{grad}_k] &= 2\delta_k^{(i}\ \mathbf{div}^{j)}\,, & [\mathbf{g}_{ij}, \mathbf{div}^k] &= -\,2\delta^k_{(i}\ \mathbf{grad}_{j)}\,,
\end{aligned} \tag{5.34}$$

and themselves obey the supersymmetry-like algebra

$$[\mathbf{div}^i, \mathbf{grad}_j] = \delta^i_j\ \Delta\,. \tag{5.35}$$

Before studying the Lie algebra cohomology of the above algebra, and the accompanying spinning particle model, in the next sections, let us briefly discuss how irreducible tensor representations can be obtained from the reducible ones depicted in (5.28).

There are two pertinent notions of irreducibility for tensors. The first is with respect to $gl(D)$ and is obtained by studying all possible permutation symmetries. This can be achieved using the operators $\mathbf{N}_i^j$ which move a box from row j to row i with a combinatorial factor equaling the number of boxes in row j, for example

$\mathbf{N}_2^3$  $= 4$  . (5.36)

Irreducible $gl(D)$ representations correspond to Young diagrams in which the number of boxes in each row decreases weakly (read from top to bottom). This amounts to tensors in the kernel of all operators

$$\{\mathbf{N}_i^{j>i}\}\,, \tag{5.37}$$

which generate the nilradical subalgebra of $gl(r)$.

For physical applications, often the stronger requirement of $so(D)$ irreducibility is placed on tensors. This amounts to additionally removing all traces and therefore tensors in the kernel of

$$\{\mathbf{N}_i^{j>i},\ \mathbf{tr}^{ij}\}\,. \tag{5.38}$$

This set generates the nilradical of $sp(2r)$. It plays an important *rôle* in the choice of first class algebra in the BRST construction of the next section.

5.4 Mixed symmetry Einstein operators

The aim of this section is to derive the generalization of the Einstein operator (5.22) to higher spins of arbitrary symmetry type. This result can actually be directly obtained by covariantizing (5.22) with respect to[4] $sp(2r)$ but here we outline its BRST derivation to connect with our path integral techniques. (For a review of existing available higher spin BRST techniques see [37], the unsymmetrized version of our Einstein operator was derived by BRST techniques recently in [38]. BRST quantization of the algebra (5.39) below was also considered in [39]).

To begin with we must choose the first class algebra generalizing (5.11). The basic philosophy is that gauging $\{\mathbf{div}^i, \mathbf{grad}_i, \Delta\}$ yields the correct differential relations on propagating physical modes-this is perhaps seen most easily in the path integral approach described in section 5.6. We also expect algebraic relations that ensure the theory describes irreducible representations of the Lorentz group. These will follow by also gauging the nilradical $sp(2r)$ operators in (5.38). Hence we study the Lie algebra cohomology of

$$\mathfrak{g} = \left\{\mathbf{tr}^{ij},\ \mathbf{div}^i,\ \mathbf{N}_i^{j>i},\ \mathbf{grad}_i,\ \Delta\right\} \tag{5.39}$$

acting on multi-symmetric forms.

The next question we must address is what degree/ghost number cohomology corresponds to the underlying physical system. This is resolved by recalling that the totally antisymmetric tensor theories possess gauge for gauge symmetries. For example, for a two-form ω with gauge transformation $\delta\omega = \boldsymbol{d}\alpha$, exact one-form gauge parameters $\alpha = \boldsymbol{d}\beta$ do not act on ω. In other words, two-forms appear at degree two in de Rham cohomology. However, multi-symmetric forms with r rows as in (5.28) can also be expanded in multi-forms of degree r or less. This implies gauge for gauge symmetries even for multi-symmetric forms and shows that we should study the degree r cohomology.

At degree r there are $\binom{\dim\mathfrak{g}}{r} = \binom{(r+1)^2}{r}$ closure relations on $\binom{\dim\mathfrak{g}}{r-1}$ fields (because here $\dim\mathfrak{g} = (r+1)^2$). These read

$$\mathfrak{g}_{[A_1}\Psi_{A_2\ldots A_{r+1}]} - \frac{r}{2} f^B_{[A_1A_2}\Psi_{|B|A_3\ldots A_{r+1}]} = 0\,. \tag{5.40}$$

These fields enjoy $\binom{\dim\mathfrak{g}}{r-1} = \binom{(r+1)^2}{r-1}$ gauge invariances following from exactness

$$\delta\Psi_{A_1\ldots A_r} = \mathfrak{g}_{[A_1}\xi_{A_2\ldots A_r]} - \frac{r-1}{2} f^B_{[A_1A_2}\xi_{|B|A_3\ldots A_r]}\,. \tag{5.41}$$

[4] We thank Stanley Deser for this observation.

Solving this system at arbitrary degree r may seem daunting, but is in fact not far more difficult than the $r = 1$, $sp(2)$ computation performed above. Let us sketch the main ideas and then give the result.

Firstly, we note that the field

$$\Psi \equiv \Psi_{\mathbf{grad}_1 \ldots \mathbf{grad}_r} = \frac{1}{r!} \epsilon^{i_1 \ldots i_r} \Psi_{\mathbf{grad}_{i_1} \ldots \mathbf{grad}_{i_r}} \tag{5.42}$$

gives the minimal covariant field content of our final detour complex. Here, and in what follows, we use a compact notation where field labels $\mathbf{grad}_i$ are soaked up with the $sl(r)$ invariant, totally antisymmetric symbol

$$\Psi^{i_1 \ldots i_n}_{\bullet} \equiv \frac{1}{(r-n)!} \, \epsilon^{i_1 \ldots i_r} \Psi_{\bullet \, \mathbf{grad}_{i_{n+1}} \ldots \mathbf{grad}_{i_r}} \, . \tag{5.43}$$

Now we turn to the closure relations (5.40). When one of the adjoint indices is Δ and all others are $\mathbf{grad}_i$ we obtain a relation analogous to the last one in (5.16)

$$\Delta \, \Psi - \mathbf{grad}_i \, \Psi^i_{\Delta} = 0 \, . \tag{5.44}$$

The field Ψ^i_{Δ} is not independent. It is eliminated by a pattern of closure relations analogous to the $sp(2)$ ones in (5.16)

$$\begin{aligned} \mathbf{div}^i \, \Psi - \mathbf{grad}_j \, \Psi^{j}_{\mathbf{div}^i} &= \Psi^i_{\Delta} \, , \\ \mathbf{tr}^{ij} \, \Psi - \mathbf{grad}_r \, \Psi^{r}_{\mathbf{tr}^{ij}} &= \Psi^{j}_{\mathbf{div}^i} + \Psi^{i}_{\mathbf{div}^j} \, . \end{aligned} \tag{5.45}$$

These imply

$$\mathbf{G} \, \Psi = \frac{1}{2} \, \mathbf{grad}_i \, \mathbf{grad}_j \, \mathbf{grad}_k \, \Psi^{k}_{\mathbf{tr}^{ij}} \, , \tag{5.46}$$

where

$$\mathbf{G} = \Delta - \mathbf{grad}_i \mathbf{div}^i + \frac{1}{2} \, \mathbf{grad}_i \mathbf{grad}_j \mathbf{tr}^{ij} \, . \tag{5.47}$$

(This operator was first derived by Labastida in [4].) As in the $sp(2)$ case, the field $\Psi^{k}_{\mathbf{tr}^{ij}}$ vanishes once we use the gauge freedom implied by exactness. However, there are still double-trace relations satisfied by the physical field Ψ. For these we consider the further closure relation

$$\mathbf{tr}^{ij} \, \Psi^{l}_{\mathbf{div}^k} - \mathbf{div}^k \, \Psi^{l}_{\mathbf{tr}^{ij}} - \mathbf{grad}_m \, \Psi^{ml}_{\mathbf{div}^k \mathbf{tr}^{ij}} = \Psi^{il}_{\mathbf{div}^k \mathbf{div}^j} + \Psi^{jl}_{\mathbf{div}^k \mathbf{div}^i} - \Psi^{kl}_{\Delta \, \mathbf{tr}^{ij}} \, . \tag{5.48}$$

This time the second and third terms on the left hand side can be gauged away as can the antisymmetric part of $\Psi^{i}_{\mathbf{div}^j}$ so, using (5.45), we must solve the equation

$$\mathbf{tr}^{ij} \, \mathbf{tr}^{kl} \Psi = \Psi^{il}_{\mathbf{div}^k \mathbf{div}^j} + \Psi^{jl}_{\mathbf{div}^k \mathbf{div}^i} - \Psi^{kl}_{\Delta \, \mathbf{tr}^{ij}} \, . \tag{5.49}$$

The antisymmetric part in k and l determines $\Psi^{kl}_{\Delta \, \mathbf{tr}^{ij}}$ in terms of $\Psi^{il}_{\mathbf{div}^k \mathbf{div}^j}$ which in turn depends on double traces of the physical field Ψ. However, symmetrizing the above equation in any three indices causes the right hand side to vanish identically. Therefore we learn the double trace relation

$$\mathbf{tr}^{i(j} \mathbf{tr}^{kl)} \Psi = 0 \, . \tag{5.50}$$

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Finally we must remember the closure relations coming from gauging the R-symmetries $\mathbf{N}_i^{j>i}$. In particular, using the trivial identity $\delta_i^{j>i}=0$ we have

$$\mathbf{N}_i^j\,\Psi-\mathbf{grad}_k\Psi^k_{\mathbf{N}_i^j}=0\,,\qquad j>i\,. \tag{5.51}$$

Gauging away the second term leaves us with the algebraic constraint

$$\mathbf{N}_i^{j>i}\,\Psi=0\,. \tag{5.52}$$

At this point there are no further constraints on the physical field Ψ and all remaining fields are either gauged away or algebraically dependent on Ψ. Let us gather together the equations of motion for Ψ;

$$\mathbf{G}\,\Psi=0=\mathbf{tr}^{i(j}\mathbf{tr}^{kl)}\,\Psi=\mathbf{N}_i^{j>i}\,\Psi\,, \tag{5.53}$$

with $\mathbf{G}$ given in (5.47). As a consistency check, it is not difficult to verify that the operators $\{\mathbf{G},\,\mathbf{tr}^{i(j}\mathbf{tr}^{kl)},\,\mathbf{N}_i^{j>i}\}$ themselves form a first class algebra.

These equations of motion are gauge invariant under transformations following from exactness (5.41)

$$\delta\Psi=\frac{1}{(r-1)!}\,\epsilon^{i_1\ldots i_r}\mathbf{grad}_{i_1}\,\xi_{i_2\ldots i_r}\equiv\mathbf{grad}_i\xi^i\,. \tag{5.54}$$

In the last term we have employed the compact notation (5.43) also for the gauge parameters. The parameters themselves are subject to constraints that can be deduced by carefully following which gauge freedoms were employed to remove all independent fields save for the physical one Ψ. These read

$$\mathbf{tr}^{(ij}\,\xi^{k)}=\mathbf{tr}^{i(j}\mathbf{tr}^{kl)}\,\xi^m=0=\mathbf{N}_i^{j>i}\,\xi^k+\delta_i^k\,\xi^{j>i}\,. \tag{5.55}$$

The longhand notation in the second term of (5.54) makes it clear that there are gauge for gauge symmetries

$$\delta\xi_{\mathbf{grad}_{i_1}\ldots\mathbf{grad}_{i_{r-1}}}=\mathbf{grad}_{[i_1}\xi_{\mathbf{grad}_{i_2}\ldots\mathbf{grad}_{i_{r-2}]}}\,, \tag{5.56}$$

coming from the Lie algebra cohomology at degree $r-1$. Of course, there are further gauge for gauge symmetries of the same form corresponding to the system being rank r when irreducible tensors are expanded in an antisymmetric basis.

To end this section, we compute the mixed symmetry analog of the Einstein operator (5.22). Firstly the field equation $\mathbf{G}\,\Psi=0$ is equivalent to

$$\mathcal{G}\,\Psi\equiv\left(1-\frac{1}{4}\,\mathbf{g}_{ij}\,\mathbf{tr}^{ij}+\frac{1}{48}\,\mathbf{g}_{ij}\mathbf{g}_{kl}\,\mathbf{tr}^{ij}\mathbf{tr}^{kl}\right)\mathbf{G}\Psi=0\,. \tag{5.57}$$

(Here, and in the formula that follows we specialize to r = 2 for simplicity). The mixed higher spin Einstein operator $\mathcal{G}$ then equals

$$\begin{aligned}\mathcal{G}\;=\;&\Delta-\mathbf{grad}_i\,\mathbf{div}^i+\frac{1}{2}\,\left(\mathbf{g}_{ij}\,\mathbf{div}^i\mathbf{div}^j+\mathbf{grad}_i\mathbf{grad}_j\,\mathbf{tr}^{ij}\right)\\&-\frac{1}{4}\,\mathbf{g}_{ij}\,\left(2\Delta-\mathbf{grad}_k\,\mathbf{div}^k\right)\,\mathbf{tr}^{ij}-\frac{1}{2}\,\mathbf{g}_{ij}\mathbf{grad}_k\,\mathbf{div}^i\mathbf{tr}^{jk}\\&+\frac{1}{48}\,\mathbf{g}_{ij}\mathbf{g}_{kl}\,(4\Delta+\mathbf{grad}_m\,\mathbf{div}^m)\,\mathbf{tr}^{ij}\mathbf{tr}^{kl}\\&-\frac{1}{6}\,\left(\mathbf{g}_{i[j}\mathbf{g}_{k]l}\,\mathbf{div}^i\mathbf{div}^j\mathbf{tr}^{kl}+\mathbf{g}_{ij}\mathbf{grad}_k\mathbf{grad}_l\,\mathbf{tr}^{i[j}\mathbf{tr}^{k]l}\right)\,.\end{aligned} \tag{5.58}$$

It is self-adjoint and obeys Bianchi and gauge invariance identities

$$\mathbf{div}^i\,\mathcal{G} \;=0 \qquad \mathrm{mod}_{\mathrm{left}}\;\mathbf{g}_{ij}\,, \tag{5.59}$$

$$\mathcal{G}\,\mathbf{grad}_i \;=0 \qquad \mathrm{mod}_{\mathrm{right}}\;\mathbf{tr}^{ij}\,, \tag{5.60}$$

ensuring the existence of a detour complex analogous to (5.26). Once again, an action principle $S=\frac{1}{2}\int(\Phi,\mathcal{G}\Phi)$ with inner product $(\cdot,\cdot)$ inherited from the underlying quantum mechanics also follows immediately. Our next task is to quantize this system. We adopt a first quantized approach which we now explain.

5.5 The $sp(2r)$ spinning particle

The particle action introduced in the previous section in equation (5.29) is invariant under global extended supersymmetry, with Sp($2r$) R-symmetry group. This action can be used to construct locally supersymmetric particle actions that give a path integral implementation of the quantum algebras studied previously.

Let us start from the phase space symplectic integral

$$S=\int dt\,\left\{p_\mu\dot{x}^\mu+iz^*_{i\mu}\dot{z}^{i\mu}\right\}\,, \tag{5.61}$$

which is invariant under the action of the global transformations with symmetry generator

$$G=\xi H+\bar{\sigma}_iS^i+\sigma^i\bar{S}_i+\frac{1}{2}\beta^{ij}\bar{K}_{ij}+\frac{1}{2}\bar{\beta}_{ij}K^{ij}+\alpha^i_jJ^j_i\,, \tag{5.62}$$

where the classical susy generators and classical $sp(2r)$ generators are, respectively, given by

$$\bar{S}_i=p\cdot z^*_i\,, \qquad S^i=p\cdot z^i\,, \qquad H=\frac{1}{2}\,p^2\,, \tag{5.63}$$

$$\bar{K}_{ij}=z^*_i\cdot z^*_j\,, \qquad J^j_i=z^*_i\cdot z^j\,, \qquad K^{ij}=z^i\cdot z^j\,. \tag{5.64}$$

Here the $u(r)$ subalgebra generated by J^j_i is made manifest. Note that the canonical quantization discussed earlier simply amounts to the replacement

$$\begin{aligned} i\bar{S}_i&\to\mathbf{grad}_i\,, & iS^i&\to\mathbf{div}^i\,, & -2H&\to\Delta\\ \bar{K}_{ij}&\to\mathbf{g}_{ij}\,, & J^j_i&\to\mathbf{N}^j_i\,, & K^{ij}&\to\mathbf{tr}^{ij}\,. \end{aligned} \tag{5.65}$$

The transformation rules for the dynamical fields can be read off from $\delta q=\{q,G\}$

$$\begin{aligned} \delta x^\mu&=\xi\,p^\mu+\bar{\sigma}_i\,z^{i\mu}+\sigma^i\,z^{*\mu}_i\,,\\ \delta p_\mu&=0\,,\\ \delta z^{i\mu}&=-i\sigma^i\,p^\mu-i\alpha^i_j\,z^{j\mu}-i\beta^{ij}\,z^{*\mu}_j\,,\\ \delta z^*_{i\mu}&=i\bar{\sigma}_i\,p^\mu+i\alpha^j_i\,z^*_{j\mu}+i\bar{\beta}_{ij}\,z^j_\mu\,. \end{aligned} \tag{5.66}$$

Gauged actions are thus obtained by adding gauge fields coupled to the above conserved charges. In particular, gauging all the global symmetries yields the action

$$S=\int dt\,\left\{p_\mu\dot{x}^\mu+iz^*_{i\mu}\dot{z}^{i\mu}-eH-\bar{s}_iS^i-s^i\bar{S}_i-\frac{1}{2}\bar{b}_{ij}K^{ij}-\frac{1}{2}b^{ij}\bar{K}_{ij}-a^i_jJ^j_i\right\}\,, \tag{5.67}$$

and the transformations for the gauge fields are obtained by requiring S to be invariant under local transformations $\Xi(t) = (\xi(t), \bar{\sigma}_i(t), \sigma^i(t), \bar{\beta}_{ij}(t), \beta^{ij}(t), \alpha^i_j(t))$, namely

$$\begin{aligned}
\delta e &= \dot{\xi} + i2\sigma^i \bar{s}_i - i2\bar{\sigma}_i s^i \,, \\
\delta s^i &= \dot{\sigma}^i + i a^i_j \sigma^j - i\alpha^i_j s^j + i\beta^{ij}\bar{s}_j - ib^{ij}\bar{\sigma}_j \,, \\
\delta \bar{s}_i &= \dot{\bar{\sigma}}_i - i a^j_i \bar{\sigma}_j + i\alpha^j_i \bar{s}_j - i\bar{\beta}_{ij} s^j + i\bar{b}_{ij}\sigma^j \,, \\
\delta b^{ij} &= \dot{\beta}^{ij} + i\left(a^i_k\,\beta^{jk} + a^j_k\,\beta^{ik}\right) - i\left(\alpha^i_k\, b^{jk} + \alpha^j_k\, b^{ik}\right) \,, \\
\delta \bar{b}_{ij} &= \dot{\bar{\beta}}_{ij} - i\left(a^k_i\,\bar{\beta}_{jk} + a^k_j\,\bar{\beta}_{ik}\right) + i\left(\alpha^k_i\,\bar{b}_{jk} + \alpha^k_j\,\bar{b}_{ik}\right) \,, \\
\delta a^i_j &= \dot{\alpha}^i_j - i\left(\alpha^i_k\, a^k_j - a^i_k\,\alpha^k_j\right) + i\left(\beta^{ik}\,\bar{b}_{jk} - b^{ik}\bar{\beta}_{jk}\right) \,.
\end{aligned} \tag{5.68}$$

However, it will be most interesting to consider partial gaugings that only involve subalgebras of the above algebras. In fact, in many of these cases it is possible to leave (part of) the abelian subgroup $\mathrm{U}(1)^r \subset \mathrm{U}(r)$ invariant, which would allow a gauge-invariant Chern-Simons action

$$S_{\mathrm{CS}} = \int dt \sum_i q_i\; a^i_i \,. \tag{5.69}$$

This can be used to fix the number of indices in a particular row of a Young diagram.

Let us single out a few interesting cases

1. *Gauge* H, $\bar{S}_i$, S^i, K^{ij}, J^i_j (or Δ, **grad**, **div**, **tr** and *all* **N**'s). This amounts to setting $b^{ij} = \beta^{ij} = 0$ in the previous transformations rules and leaves in particular

$$\delta a^i_j = \dot{\alpha}^i_j - i\left(\alpha^i_k\, a^k_j - a^i_k\,\alpha^k_j\right) \tag{5.70}$$

   from which it is obvious that $a = \sum_i^r a^i_i$ is invariant and the unique Chern-Simons term is $S_{\mathrm{CS}} = \int dt\; q\, a$.

2. *Gauge* H, $\bar{S}_i$, S^i, K^{ij}, $J^{j\geq i}_i$ (or Δ, **grad**, **div**, **tr** and nilradical **N**'s). This amounts to setting $b^{ij} = \beta^{ij} = 0$ and $a^i_j = \alpha^i_j = 0$ when $i > j$, from which

$$\delta a^i_i = 0 \,, \quad \text{(no sum implied)} \tag{5.71}$$

   and allowed Chern-Simons are given by $S_{\mathrm{CS}} = \int dt\;\; \sum_i q_i\, a^i_i$.

We are now ready to quantize these gauged models.

5.6 Counting degrees of freedom

In the present section we use the particle actions described above to compute the number of degrees of freedom for mixed higher spin tensor multiplets. We study the partition function

$$Z \sim \int_{S^1} \frac{DXDE}{\mathrm{Vol(Gauge)}}\, e^{iS[X,E] + iS_{\mathrm{CS}}[E]} \,, \tag{5.72}$$

where X collectively denotes all dynamical fields, whereas E denotes gauge fields. In the latter we need to carefully gauge fix all the gauge symmetries present in the spinning

particle action. We now Wick rotate to Euclidean time (periodic boundary conditions) and use the Faddeev-Popov trick to extract the volume of the gauge group to set a gauge choice that completely fixes all the supergravity fields up to some constant moduli fields

$$E = (e, \bar{s}_i, s^i, \bar{b}_{ij}, b^{ij}, a^i{}_j) = (\beta, 0, 0, 0, 0, \theta_i \delta^i_j)\,, \tag{5.73}$$

where $\hat{a}^i{}_j = \theta_i \delta^i_j$ is the most generic constant element of the Cartan subalgebra of $sp(2r)$, with θ_i being angles taking values in a fundamental domain. We thus have

$$Z = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta} \int \frac{d^D x}{(2\pi\beta)^{D/2}}\, \mathrm{DoF}(q,r) \tag{5.74}$$

with

$$\begin{aligned}
\mathrm{DoF}(q,r) \equiv K_r \; & \underbrace{\prod_{i=1}^r \left[\int_0^{2\pi} \frac{d\theta_i}{2\pi}\, e^{iq_i\theta_i}\right]}_{\text{Cartan moduli for Sp}(2r)+\text{CS}} \underbrace{\prod_{i=1}^r \Big[\mathrm{Det}(\partial_\tau - i\theta_i)\Big]}_{\bar{S}_i} \underbrace{\prod_{i=1}^r \Big[\mathrm{Det}(\partial_\tau + i\theta_i)\Big]}_{S^i} \\
& \times \underbrace{\prod_{i=1}^r \Big[\mathrm{Det}(\partial_\tau + i\theta_i)\Big]^{-D}}_{z,z^*} \underbrace{\prod_{i=1}^r \Big[\mathrm{Det}(\partial_\tau - 2i\theta_i)\Big]}_{\bar{K}_{ii}\ \text{no sum}} \underbrace{\prod_{i=1}^r \Big[\mathrm{Det}(\partial_\tau + 2i\theta_i)\Big]}_{K^{ii}\ \text{no sum}} \\
& \times \underbrace{\prod_{i\neq j} \Big[\mathrm{Det}\Big(\partial_\tau - i(\theta_i - \theta_j)\Big)\Big]}_{\text{U}(r)\ \text{step operator}} \underbrace{\prod_{i<j} \Big[\mathrm{Det}\Big(\partial_\tau - i(\theta_i + \theta_j)\Big)\Big]}_{\bar{K}_{ij},\ i\neq j} \\
& \times \underbrace{\prod_{i<j} \Big[\mathrm{Det}\Big(\partial_\tau + i(\theta_i + \theta_j)\Big)\Big]}_{K^{ij},\ i\neq j}
\end{aligned} \tag{5.75}$$

being the number of degrees of freedom, and K_r^{-1} the number of fundamental domains included in the integration domain. All the determinants are evaluated with periodic boundary conditions because the fields traced over are bosonic.

The latter expression should really be understood as a generating function, including all ingredients for all possible $sp(2r)$ particle actions. Clearly, for each specific gauged action, one has to pick out only those determinants that are involved in its gauge fixing. Let us consider the example 2 of section 5.5, which we claim corresponds to the BRST cohomology of the algebra (5.39) computed in section 5.4. For that theory we clearly have

$$\begin{aligned}
\mathrm{DoF}(q,r) = K_r \; & \prod_{i=1}^r \left[\int_0^{2\pi} \frac{d\theta_i}{2\pi}\, e^{iq_i\theta_i}\right] \prod_{i=1}^r \Big[\mathrm{Det}(\partial_\tau + i\theta_i)\Big]^{2-D} \prod_{i=1}^r \Big[\mathrm{Det}(\partial_\tau - 2i\theta_i)\Big] \\
& \times \prod_{i<j} \Big[\mathrm{Det}\Big(\partial_\tau - i(\theta_j - \theta_i)\Big)\Big] \prod_{i<j} \Big[\mathrm{Det}\Big(\partial_\tau + i(\theta_i + \theta_j)\Big)\Big]\,.
\end{aligned} \tag{5.76}$$

Using that

$$\mathrm{Det}(\partial_\tau + i\theta) = 2i\sin(\theta/2) \tag{5.77}$$

and making the change of variables $w_i = e^{-i\theta_i}$ we obtain

$$\mathrm{DoF}(q,r) = K_r \prod_{i=1}^{r} \left[-\oint \frac{dw_i}{2\pi i w_i} \frac{(1-w_i)^{2-D}(w_i^2-1)}{w_i^{q_i+r+1-D/2}} \right] \prod_{i<j}(w_j - w_i)(1-w_j w_i) \tag{5.78}$$

that gives

$$\begin{aligned} \mathrm{DoF}(q,r) \;=\; & K_r \prod_{i=1}^{r} \left[-\frac{1}{n_i!} \frac{d^{n_i}}{dw_i^{n_i}} \right] \\ & \times \prod_{i=1}^{r} (1-w_i)^{2-D}(w_i^2-1) \prod_{i<j}(w_j - w_i)(1-w_j w_i) \Big|_{w_i=0} \end{aligned} \tag{5.79}$$

with $n_i = q_i + r + 1 - D/2$.

Let us consider a few specific examples. The simplest one is clearly $r = 1$ for which we set $s \equiv q + 2 - D/2 = n$ and, using that $K_1 = 1$, we obtain

$$\mathrm{DoF}(s,1) = \frac{D+2s-4}{s} \begin{pmatrix} D+s-5 \\ s-1 \end{pmatrix} \tag{5.80}$$

which is precisely the dimension of a Young tableau of $so(D-2)$ with 1 row and s columns.

For $r = 2$, we identify $n_2 \equiv s_2 + 1$ and $n_1 \equiv s_1$, then $K_2 = 1$ and using (5.79) we obtain

$$\begin{aligned} \mathrm{DoF}(s_2,s_1,2) \;=\; & \frac{(D+s_1-7)!(D+s_2-6)!(s_2-s_1+1)!}{(D-6)!(D-4)!(s_2+1)!\,s_1!\,(s_2-s_1)!} \\ & \times (D+s_2+s_1-5)(D+2s_2-4)(D+2s_1-6)\,. \end{aligned} \tag{5.81}$$

This is the dimension of a Young tableau with s_2 boxes in the first row and $s_1 \leq s_2$ boxes in the second row.

For arbitrary r we identify $n_k = s_k + k - 1$ so that (5.79) should yield the dimension of a generic Young tableau with s_k boxes in the $k-$th row and $s_1 \leq s_2 \leq \cdots \leq s_r$. In all these cases, the physical degrees of freedom correspond then to irreducible $so(D-2)$ representations. In fact, for arbitrary r, we expect that equation (5.79) is the generating function for the dimensions of irreducible $so(D-2)$ representations. In the next section we obtain the same result from the second quantized equations of motion that follow from the BRST cohomology computation of section 5.4.

5.7 Lightcone degrees of freedom

Our final computation is to verify that the path integral degree of freedom counts match those obtained by a direct analysis of the second quantized field equations (5.53) which we reproduce here for convenience

$$\left(\Delta - \mathbf{grad}_i \mathbf{div}^i + \frac{1}{2}\mathbf{grad}_i \mathbf{grad}_j \mathbf{tr}^{ij}\right)\Psi = 0\,, \tag{5.82}$$

$$\mathbf{tr}^{i(j}\mathbf{tr}^{kl)}\,\Psi = 0\,, \tag{5.83}$$

$$\mathbf{N}_i^{j>i}\,\Psi = 0\,. \tag{5.84}$$

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These equations enjoy the gauge invariances

$$\delta\Psi = \mathbf{grad}_i \xi^i \,, \quad \text{where } \mathbf{tr}^{(ij}\,\xi^{k)} = \mathbf{tr}^{i(j}\mathbf{tr}^{kl)}\,\xi^m = 0\,. \tag{5.85}$$

A detailed proof of these dimension counts, which are essentially just a higher dimensional analog of Wigner's original computation of unitary representations of the Poincaré group [40], were only given rather recently in [41]. A BRST lightcone version of this computation was given in [38]. By far the the speediest method to perform this computation, however, is to employ lightcone gauge directly to the second quantized Lagrangian. For completeness we present that result here. Expressing the metric as

$$ds^2 = 2dx^+dx^- + d\vec{x}^{\,2}\,, \tag{5.86}$$

we assume that $\partial/\partial x^-$ is invertible and set it equal to unity in what follows.

Our main philosophy is to expand fields and field equations in powers of differentials $d_i x^-$ in the x^- direction. All the operators $(\Delta, \mathbf{grad}_i, \mathbf{div}^i, \mathbf{g}_{ij}, \mathbf{N}^j_i, \mathbf{tr}^{ij})$ then have $(D-2)$-dimensional analogs operating in the $\vec{x}$ directions. We denote these by hats so that

$$\begin{aligned}
\Delta &= 2\,\frac{\partial}{\partial x^+} \;+\; \widehat{\Delta}\,,\\
\mathbf{grad}_i &= d_i x^- + d_i x^+ \frac{\partial}{\partial x^+} + \widehat{\mathbf{grad}}_i\,,\\
\mathbf{div}^i &= \frac{\partial}{\partial(d_i x^-)}\,\frac{\partial}{\partial x^+} + \frac{\partial}{\partial(d_i x^+)} + \widehat{\mathbf{div}}^i\,,\\
\mathbf{g}_{ij} &= 2\,d_{(i}x^- d_{j)}x^+ + \widehat{\mathbf{g}}_{ij}\,,\\
\mathbf{N}^j_i &= d_i x^- \frac{\partial}{\partial(d_j x^-)} + d_i x^+ \frac{\partial}{\partial(d_j x^+)} + \widehat{\mathbf{N}}^j_i\,,\\
\mathbf{tr}^{ij} &= 2\frac{\partial}{\partial(d_{(i}x^-)}\,\frac{\partial}{\partial(d_{j)}x^+)} + \widehat{\mathbf{tr}}^{ij}\,.
\end{aligned} \tag{5.87}$$

Now we decompose the field Ψ as

$$\Psi(d_i x^-) = \psi + d_i x^- \chi^i\,, \tag{5.88}$$

where ψ is independent of $d_i x^-$. On the other hand the fields χ^i are $d_i x^-$ dependent, and we focus on the term with the highest power of $d_i x^-$. Examining the terms of highest order in the field equation (5.82) coming from the $\mathbf{grad}^2\,\mathbf{tr}$-term, we see that the highest order term in χ^i is lightcone symmetric-trace-free (i.e., annihilated by $\widehat{\mathbf{tr}}^{(ij}$). However, from the lightcone decomposition of $\mathbf{grad}_i$ in (5.87), the gauge invariance (5.85) becomes

$$\delta\Psi = \Big(d_i x^- + \cdots\Big)\xi^i\,. \tag{5.89}$$

The trace condition on the gauge parameter ξ^i exactly ensures that its highest $d_i x^-$ term is also lightcone symmetric-trace-free. Hence we may algebraically gauge away the highest order term in χ^i. Iterating the above argument allows us to gauge away all of χ^i so that

$$\Psi = \psi(d_i\vec{x}, d_i x^+)\,. \tag{5.90}$$

Our computation is completed by solving (5.82) for fields of the above form. Again we work order by order in $d_i x^-$. At highest order we learn

$$\widehat{\mathbf{tr}}^{ij}\,\psi = 0\,. \tag{5.91}$$

To study lower order terms we split

$$\psi(d_i\vec{x}, d_i x^+) = \widehat{\psi}(d_i\vec{x}) + d_i x^+ \psi^i\,, \tag{5.92}$$

where the field $\widehat{\psi}$ only has $(D-2)$-dimensional indices. The fields ψ^i are all dependent because the next to leading order terms in (5.82) imply

$$\frac{\partial\psi}{\partial(d_i x^+)} + \widehat{\mathbf{div}}_i\psi = 0\,. \tag{5.93}$$

This condition can always be solved in terms of the ψ_i in (5.92) so it remains to gather the remaining lowest order terms in (5.82) which read

$$\left\{2\frac{\partial}{\partial x^+} + \widehat{\Delta}\right\}\widehat{\psi} = 0\,. \tag{5.94}$$

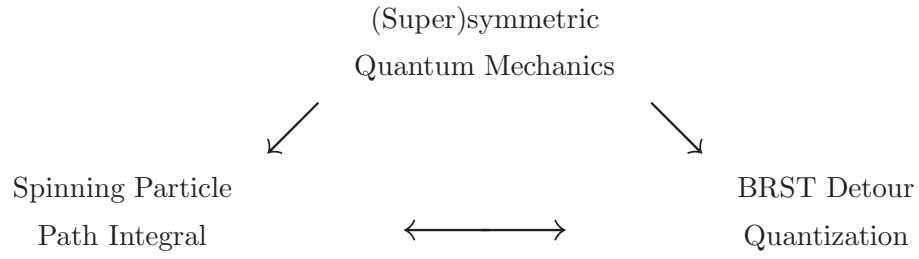
This is simply the D-dimensional Klein-Gordon equation. Finally we still need to impose the symmetry condition. It is not hard to see that it implies

$$\widehat{\mathbf{N}}_i^{j>i}\,\widehat{\psi} = 0\,. \tag{5.95}$$

Hence the independent light cone degree of freedom are described by a totally symmetric $(D-2)$-dimensional tensors, which solve the D-dimensional wave equation and are both $(D-2)$-dimensional trace-free and irreducible. Or in other words, the degree of freedom count is given by dimensions of $so(D-2)$ irreducible representations. This shows the claimed equivalence between BRST and path integral quantizations.

6 Conclusions

In this Article we have tackled the problem of constructing and quantizing quantum field theories for tensor fields with general symmetry types using a worldline approach. As depicted below, our starting point was a quantum mechanical (super)symmetric model whose wave functions are the type of tensor fields appearing in the desired second quantized model.



Thereafter we identified first class constraint operator algebras acting on the quantum mechanical Hilbert space. From these algebras one can build a first quantized gauge theory in two ways, either path integral methods, or BRST quantization. The former led to a path integral representation of a spinning particle model while the latter, using the detour complex idea, yielded the classical equations of motion of a second quantized gauge field theory. The path integral approach gave a worldline method for computing quantum quantities in the second quantized field theory, the simplest of which was a count of the physical degrees of freedom. These can of course also be computed by studying the dimensionality of the Cauchy data of the classical field equations. Indeed we found that these two methods gave identical answers for a large class of higher spin theories.

The quantum mechanical models we used as a starting point fall into a very broad class of models labeled by their R-symmetry groups which are given by general orthosymplectic supergroups. In that language our Article focused on the $osp(2|0)$ and $osp(0|2r)$ models. However, it is clear that our methods can be generalized to any of the $osp(Q|2r)$ models. When Q is odd, these models describe spinor-tensor fields in second quantization. The case $osp(2|1)$ has been studied in [26] to describe spinor valued totally symmetric tensor theories, but clearly a complete description of fermionic second quantized models would be desirable.

There are many other directions our results lead to. The most interesting of course, would be to shed light on self-interactions of higher spin fields. By now a large literature exists on this subject, a consistent theme being that interactions for higher spins requires towers of infinitely many second quantized fields (see [8] for an extensive review of these developments). In simplest terms this points at a difficulty gauging the number operator(s) $\mathbf{N}$ of our quantum mechanical models. From the path integral viewpoint, this difficulty can be seen through the sparsity of consistent world-line Chern-Simons terms that can be added to the worldline action.

There are two other most interesting, and in fact related, applications of our results. These are computations of higher (second quantized) quantum amplitudes and interactions with backgrounds. Higher amplitudes are encoded, for example, by studying the dependence of the worldline effective action on *arbitrary* background fields. It is not difficult to couple our underlying quantum mechanical models to either background Yang-Mills or gravitational fields by twisting the connection appearing in the covariant canonical momentum operator. However in general this can produce obstructions to our constraint algebras being first class. These obstructions have been studied and explicated in [36]. The phenomenon of higher spin fields suffering inconsistencies in backgrounds is one that has been known for a long time (dating back to work on coupling massive spin 3/2 fields, see for a thorough account [42]).

It is possible to view these obstructions to first class algebras in general spaces in a more positive light. Namely, these algebras can be used to develop powerful invariants for determining the underlying geometry of the background manifold. In turn, when the background manifold belongs to a special class of geometries, consistency and even enhanced symmetries and constraint algebras can result. The special *rôle* played by certain geometries in string theory is an example of this phenomenon. Another example are the Kähler

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higher spin models constructed in [43–45] and (p, q)-form Kähler electromagnetism [46]. The latter of these theories follows from the detour construction [47]. It would be most interesting to compute its path integral quantization.

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